MATH 365 Homework #5

Due: Thursday, Mar. 1, 2012

Review of "Gaussian Elimination Method"

1. Principle

Given a square matrix $A = (a_{i,j})_{1\le i,j\le n} \in \mathbb{R}^{n\times n}$ and a vector $b = (b_i)_{1\le i\le n} \in \mathbb{R}^n$, solve the following linear system $Ax = b$ for the unknown vector $x = (x_i)_{1\le i\le n} \in \mathbb{R}^n$ in terms of Gaussian elimination method as follows:

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(1). At step k ($k = 1, \cdots, n-1$), suppose the matrix $A^k = (a^k_{i,j})_{1\le i,j\le n}$ at the $(k-1)$th step has been obtained from the original coefficient matrix A $(= A^1)$ with its first $(k-1)$ columns having zeros for all entries below its diagonal.

If $a^k_{k,k} \neq 0$, then we select $\widetilde{a}^k_{k,k} = a^k_{k,k}$ as the pivot of the kth step. Otherwise, we look for $a^k_{i,k} \neq 0$ $(i \ge k+1)$ in the same kth column, and swap the kth row with the ith row such that $\widetilde{a}^k_{k,k} = a^k_{i,k} \neq 0$. By doing such pivoting we obtain a new matrix $\widetilde{A}^k = (\widetilde{a}^k_{i,j})_{1\le i,j\le n}$ and select $\widetilde{a}^k_{k,k}$ as the pivot of the kth step.

Now we convert $\widetilde{A}^k$ to $A^{k+1} = (a^{k+1}_{i,j})_{1\le i,j\le n}$ by means of the following formula

$$\begin{cases} a^{k+1}_{i,j} = \widetilde{a}^i_{i,j}, & 1 \le i \le k, i \le j \le n; \\ a^{k+1}_{i,j} = \widetilde{a}^k_{i,j} - \frac{\widetilde{a}^k_{i,k}}{\widetilde{a}^k_{k,k}}\widetilde{a}^k_{k,j}, & k+1 \le i, j \le n; \\ a^{k+1}_{i,j} = 0, & \text{elsewhere.} \end{cases}$$

Simultaneously, the corresponding right-hand-side vector $b^k = (b^k_i)_{1\le i\le n}$ obtained at the $(k-1)$th step is converted to $\widetilde{b}^k = (\widetilde{b}^k_i)_{1\le i\le n}$ due to a possible pivoting, and further converted to $b^{k+1} = (b^{k+1}_i)_{1\le i\le n}$ by the following formula

$$\begin{cases} b^{k+1}_i = \widetilde{b}^i_i, & 1 \le i \le k; \\ b^{k+1}_i = \widetilde{b}^k_i - \frac{\widetilde{a}^k_{i,k}}{\widetilde{a}^k_{k,k}}\widetilde{b}^k_k, & k+1 \le i \le n. \end{cases}$$

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(2). At step $(n-1)$, we eventually obtain the upper triangular matrix A^n and the corresponding right-hand-side vector b^n. Then we use backward substitution method to find the unknown vector $x = (x_i)_{1 \le i \le n}$ using the following formula

$$x_i = \frac{1}{a_{i,i}^n}(b_i^n - \sum_{j=i+1}^{n} a_{i,j}^n x_j), \quad n \ge i \ge 1.$$

2. Pseudo-code

<u>Pseudo-code of Gaussian Elimination Method</u>

```
function x = gauss(A,b)
n = length(b)

% Forward elimination
for k=1 to n-1

% Select the pivot
    p = k
    while A(p,k) = 0 and p ≤ n
        p = p+1
    end while
    if p ≠ k
        swap the p-th row with the k-th row
    end if

    for i=k+1 to n
        xmult = A(i,k)/A(k,k)
        for j=k+1 to n
            A(i,j) = A(i,j)-xmult*A(k,j)
        end j
        b(i) = b(i)-xmult*b(k)
    end i
end k

% Backward substitution
x(n) = b(n)/A(n,n)
for i=n-1 to 1 with increment ‘‘-1’’
    sum = b(i)
    for j=i+1 to n
        sum = sum-A(i,j)*x(j)
    end j
    x(i) = sum/A(i,i)
end i

end function
```

Review of "LU Decomposition Method"

1. Principle

Given a square matrix $A = (a_{i,j})_{1\le i,j\le n} \in \mathbb{R}^{n\times n}$ and a vector $b = (b_i)_{1\le i\le n} \in \mathbb{R}^n$, solve the following linear system $Ax = b$ for the unknown vector $x = (x_i)_{1\le i\le n} \in \mathbb{R}^n$ using LU Decomposition method as follows:

$$Ax = b \text{ and } A = LU \Leftrightarrow L(Ux) = b \Leftrightarrow Ly = b \text{ and } Ux = y,$$

where

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}, L = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ l_{2,1} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ l_{n,1} & \cdots & l_{n,n-1} & 1 \end{pmatrix}, U = \begin{pmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,n} \\ 0 & u_{2,2} & \cdots & u_{2,n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{n,n} \end{pmatrix}.$$

Thereafter, we solve $Ly = b$ for y first using forward substitution, and then solve $Ux = y$ for x using backward substitution, eventually. The entire algorithms of LU Decomposition method are given as follows

$$a_{i,j} = \sum_{k=1}^{\min(i,j)} l_{i,k}u_{k,j}.$$

(1). Step 1: We deal with Column 1 of L and U by fixing $j = 1$ and vary i:

$$\begin{array}{lll} a_{1,1} = l_{1,1}u_{1,1} & \Rightarrow & u_{1,1} = a_{1,1}; \\ a_{2,1} = l_{2,1}u_{1,1} & \Rightarrow & l_{2,1} = \frac{a_{2,1}}{a_{1,1}}; \\ \vdots & & \vdots \\ a_{n,1} = l_{n,1}u_{1,1} & \Rightarrow & l_{n,1} = \frac{a_{n,1}}{a_{1,1}}. \end{array} \tag{1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

(2). Step j ($j = 2, \cdots, n$): We assume that we have computed the first $(j-1)$ columns of L and U. Then we read the jth column of A:

$$\begin{array}{lll} a_{1,j} = l_{1,1}u_{1,j} & \Rightarrow & u_{1,j} = a_{1,j}; \\ a_{2,j} = l_{2,1}u_{1,j} + l_{2,2}u_{2,j} & \Rightarrow & u_{2,j} = a_{2,j} - l_{2,1}u_{1,j}; \\ \vdots & & \vdots \\ a_{j,j} = l_{j,1}u_{1,j} + \cdots + l_{j,j-1}u_{j-1,j} + l_{j,j}u_{j,j} & \Rightarrow & u_{j,j} = a_{j,j} - \sum\limits_{k=1}^{j-1} l_{j,k}u_{k,j}; \\ a_{j+1,j} = l_{j+1,1}u_{1,j} + \cdots + l_{j+1,j-1}u_{j-1,j} + l_{j+1,j}u_{j,j} & \Rightarrow & l_{j+1,j} = \frac{a_{j+1,j} - \sum_{k=1}^{j-1} l_{j+1,k}u_{k,j}}{u_{j,j}}; \\ \vdots & & \vdots \\ a_{n,j} = l_{n,1}u_{1,j} + \cdots + l_{n,j-1}u_{j-1,j} + l_{n,j}u_{j,j} & \Rightarrow & l_{n,j} = \frac{a_{n,j} - \sum_{k=1}^{j-1} l_{n,k}u_{k,j}}{u_{j,j}}, \end{array} \tag{2}$$

Note the division by the pivot $u_{j,j}$, which should not be zero due to the nonsingularity of all of diagonal submatrices of order k $(1 \leq k \leq n)$ of A.

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(3). Step (n+1): Forward substitution for solving $Ly = b$:

$$y_i = b_i - \sum_{j=1}^{i-1} l_{i,j} y_j, \quad (i = 1, 2, \cdots, n). \tag{3}$$

(4). Step (n+2): Backward substitution for solving $Ux = y$:

$$x_i = \frac{1}{u_{i,i}}(y_i - \sum_{j=i+1}^{n} u_{i,j} x_j), \quad (i = n, n-1, \cdots, 1). \tag{4}$$

2. Pseudo-code

Pseudo-code of LU Factorization

```
for k=1 to n-1
    for i=k+1 to n
        a(i,k)=a(i,k)/a(k,k)
        for j=k+1 to n
            a(i,j)=a(i,j)-a(i,k)a(k,j)
        end j
    end i
end k
```

Homework Problems

Given the following coefficient matrix

$$A = \begin{pmatrix} 2.0373 & 0.2961 & 1.4826 & 0.0915 & 2.4165 & 2.9361 & 1.5648 & 2.919 \\ 1.1865 & 0.7857 & 2.3373 & 2.2323 & 1.7301 & 2.1381 & 0.2901 & 1.947 \\ 1.1022 & 1.0062 & 2.145 & 1.5 & 0.5487 & 1.5015 & 2.4543 & 2.4009 \\ 2.964 & 2.0391 & 2.7111 & 1.4397 & 0.7197 & 1.4133 & 2.4525 & 1.3614 \\ 0.1131 & 0.4098 & 2.6727 & 2.7141 & 2.6595 & 0.1788 & 2.1672 & 1.2972 \\ 2.6556 & 2.1636 & 1.0026 & 1.8297 & 0.0861 & 2.046 & 0.4497 & 2.4759 \\ 2.7399 & 0.3204 & 2.0961 & 1.8531 & 1.4697 & 0.1272 & 1.9788 & 0.2505 \\ 2.3886 & 1.9614 & 0.5934 & 2.5782 & 0.5037 & 0.2142 & 1.5558 & 0.3996 \end{pmatrix},$$

and the right-hand-side vector

$$b = \begin{pmatrix} 9.5974 \\ 3.4039 \\ 5.8527 \\ 2.2381 \\ 7.5127 \\ 2.5510 \\ 5.0596 \\ 6.9908 \end{pmatrix}.$$

1. Based on the pseudo-code of "Gaussian Elimination Method", **write and run a real source code on the computer** to

   (1.1) (20 points) solve the linear system $Ax = b$.

   (1.2) (10 points) verify whether the obtained vector x in question (1.1) is the solution of $Ax = b$.

   (1.3) (20 points) compute $\det(A)$ using Gaussian Elimination formula:

   $$\det(A) = (-1)^m \det(A^n) = (-1)^m \prod_{i=1}^{n} a_{i,i}^n = (-1)^m a_{1,1}^n a_{2,2}^n \cdots a_{n,n}^n,$$

   where A^n is the final upper triangular matrix defined in the above principle of Gaussian Elimination Method, m is the number of permutation (row swapping).

   Verify your result by comparing **det(A)** with **det(B)** that is obtained by the method of "Determinant Expansion by Minors" in Homework #3, where $A = B$.

2. Based on the LU Decomposition algorithm/formula Eqns. (1)-(4) in its principle at above, write and run a real source code on the computer to

   (2.1) (15 points) find out the corresponding triangular matrices L and U for A.

   (2.2) (10 points) solve this linear system $Ax = b$ for x.

3. Based on the pseudo-code of LU factorization at above, write and run a real source code on the computer to

   (3.1) (15 points) find out the corresponding triangular matrices L and U for A, and compare your results with the ones obtained in Question (2.1).

   (3.2) (10 points) compute $\det(A)$ via the upper triangular matrix U, and compare your result with the one obtained in Question (1.3).

Note: In your homework, the right source code and the correct output results must be shown altogether in order to obtain full score.