MATH 365 Homework #6

Due: Thursday, Mar. 8, 2012

Review of "Cholesky Decomposition Method"

1. Principle

Given a real symmetric positive definite square matrix $A = (a_{i,j})_{1\le i,j\le n} \in \mathbb{R}^{n\times n}$ and a vector $b = (b_i)_{1\le i\le n} \in \mathbb{R}^n$, solve the following linear system $Ax = b$ for the unknown vector $x = (x_i)_{1\le i\le n} \in \mathbb{R}^n$ using Cholesky Decomposition method as follows:

$$Ax = b \text{ and } A = BB^t \Leftrightarrow B(B^t x) = b \Leftrightarrow By = b \text{ and } B^t x = y,$$

where

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}, B = \begin{pmatrix} b_{1,1} & 0 & \cdots & 0 \\ b_{2,1} & b_{2,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ b_{n,1} & \cdots & b_{n,n-1} & b_{n,n} \end{pmatrix}, B^t = \begin{pmatrix} b_{1,1} & b_{2,1} & \cdots & b_{n,1} \\ 0 & b_{2,2} & \ddots & b_{n,2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{n,n} \end{pmatrix},$$

here B is a lower triangular matrix with strictly positive diagonal entries, i.e. $b_{i,i} > 0\,(i = 1, \cdots, n)$. Thus, its transpose matrix B^t is a upper triangular matrix with strictly positive diagonal entries as well.

Thereafter, we solve $By = b$ for y first using forward substitution, and then solve $B^t x = y$ for x using backward substitution, eventually. The entire algorithms of Cholesky Decomposition method are given as follows

$$a_{i,j} = \sum_{k=1}^{\min(i,j)} b_{i,k} b_{j,k}.$$

(1). Step 1: We deal with Column 1 of B by fixing $j = 1$ and vary i:

$$\begin{array}{lll} a_{1,1} = b_{1,1}^2 & \Rightarrow & b_{1,1} = \sqrt{a_{1,1}}; \\ a_{2,1} = b_{2,1} b_{1,1} & \Rightarrow & b_{2,1} = \frac{a_{2,1}}{b_{1,1}}; \\ \vdots & & \vdots \\ a_{n,1} = b_{n,1} b_{1,1} & \Rightarrow & b_{n,1} = \frac{a_{n,1}}{b_{1,1}}. \end{array} \tag{1}$$

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(2). Step j ($j = 2, \cdots, n-1$): We assume that we have computed the first $(j-1)$ columns of B. Then we read the jth column of A:

$$
\begin{array}{lll}
a_{j,j} = b_{j,1}^2 + \cdots + b_{j,j-1}^2 + b_{j,j}^2 \quad (j = 2, \cdots, n) & \Rightarrow & b_{j,j} = \sqrt{a_{j,j} - \sum\limits_{k=1}^{j-1} b_{j,k}^2}; \\
a_{j+1,j} = b_{j+1,1}b_{j,1} + \cdots + b_{j+1,j-1}b_{j,j-1} + b_{j+1,j}b_{j,j} & \Rightarrow & b_{j+1,j} = \frac{a_{j+1,j} - \sum\limits_{k=1}^{j-1} b_{j+1,k}b_{j,k}}{b_{j,j}}; \\
\vdots & & \vdots \\
a_{i,j} = b_{i,1}b_{j,1} + \cdots + b_{i,j-1}b_{j,j-1} + b_{i,j}b_{j,j} & \Rightarrow & b_{i,j} = \frac{a_{i,j} - \sum\limits_{k=1}^{j-1} b_{i,k}b_{j,k}}{b_{j,j}} \ (i = j+1, \cdots, n); \\
\vdots & & \vdots \\
a_{n,j} = b_{n,1}b_{j,1} + \cdots + b_{n,j-1}b_{j,j-1} + b_{n,j}b_{j,j} & \Rightarrow & b_{n,j} = \frac{a_{n,j} - \sum\limits_{k=1}^{j-1} b_{n,k}b_{j,k}}{b_{j,j}},
\end{array} \tag{2}
$$

Note the division by the pivot $b_{j,j}$, which is positive and thus nonzero due to the definition of matrix B.

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(3). Step (n+1): Forward substitution for solving $By = b$:

$$
y_i = \frac{b_i - \sum\limits_{j=1}^{i-1} b_{i,j}y_j}{b_{i,i}}, \quad (i = 1, 2, \cdots, n). \tag{3}
$$

(4). Step (n+2): Backward substitution for solving $B^t x = y$:

$$
x_i = \frac{(y_i - \sum\limits_{j=i+1}^{n} b_{j,i}x_j)}{b_{i,i}}, \quad (i = n, n-1, \cdots, 1). \tag{4}
$$

2. Pseudo-code of Cholesky Factorization

<u>`Version 1 using lower triangular matrix L`</u>

```
for k=1 to n-1
    l(k,k)=√a(k,k)
    for m=k+1 to n
        l(m,k)=a(m,k)/l(k,k)
        for j=k+1 to n
            for i=j to n
                a(i,j)=a(i,j)-l(i,k)l(j,k)
            end i
```

```
        end j
    end m
end k
l(n,n)=√a(n,n)
```

Version 2 using original matrix A

```
for j=1 to n
    for k=1 to j-1
        a(j,j)=a(j,j)-a(j,k)a(j,k)
    end k
    a(j,j)=√a(j,j)
    for i=j+1 to n
        for k=1 to j-1
            a(i,j)=a(i,j)-a(j,k)*a(i,k)
        end k
        a(i,j)=a(i,j)/a(j,j)
    end i
end j
```

Homework Problems

Given the following coefficient matrix

$$A = \begin{pmatrix} 35.836 & 20.0433 & 26.3189 & 25.1571 & 15.4733 & 20.6763 & 24.0837 & 23.301 \\ 20.0433 & 14.6742 & 15.0619 & 16.9476 & 6.8295 & 11.9028 & 13.7083 & 14.3384 \\ 26.3189 & 15.0619 & 32.5065 & 26.9765 & 21.3285 & 19.3257 & 26.2256 & 24.4308 \\ 25.1571 & 16.9476 & 26.9765 & 30.1095 & 17.3402 & 14.3453 & 22.386 & 19.7201 \\ 15.4733 & 6.8295 & 21.3285 & 17.3402 & 19.1458 & 13.5818 & 16.8893 & 16.952 \\ 20.6763 & 11.9028 & 19.3257 & 14.3453 & 13.5818 & 21.7242 & 14.2585 & 23.6775 \\ 24.0837 & 13.7083 & 26.2256 & 22.386 & 16.8893 & 14.2585 & 25.8063 & 19.4059 \\ 23.301 & 14.3384 & 24.4308 & 19.7201 & 16.952 & 23.6775 & 19.4059 & 27.9643 \end{pmatrix},$$

and the right-hand-side vector

$$b = \begin{pmatrix} 9.5974 \\ 3.4039 \\ 5.8527 \\ 2.2381 \\ 7.5127 \\ 2.5510 \\ 5.0596 \\ 6.9908 \end{pmatrix}.$$

1. Based on any version of the pseudo-code of "Cholesky Decomposition Method", **write and run a real source code on the computer** to

   (1.1) (20 points) find out the corresponding lower triangular matrices B for A such that $A = BB^t$.

   (1.2) (20 points) solve this linear system $Ax = b$ for x using Cholesky decomposition method, and verify your solution using $Ax - b$.

   (1.3) (10 points) compute $\det(A)$ via the lower triangular matrix B, and then recompute $\det(A)$ using "LU Decomposition Method" done in Homework #5. Compare your results to see if they are the same.

2. (20 points) Given a linear system $Ax = b$ where A is a nonsingular real square matrix, describe the principle of QR Decomposition method by indicating the form of matrices Q and R and converting the original linear system to a simpler linear system which is easy to solve. The decomposition algorithm and formula are not required.

3. (15 points) Given a orthogonal matrix Q, find out every possible value of the determinant of Q.

4. (15 points) Suppose matrix A has its QR factorization, find out every possible value of the determinant of A.

Note: In your homework, the right source code and the correct output results must be shown altogether in order to obtain full score.